Periodic orbits of boundary logistic map and new kind of modified Chebyshev polynomials

Edyta Hetmaniok, Piotr Lorenc, Damian Słota
and Roman Wituła

Abstract. In the paper we discuss the periodic orbits of maps connected with the boundary logistic map. In consequence some new kind of the modified Chebyshev polynomials is defined and intensively studied. Many fundamental relations for these polynomials are presented and discussed. Concepts of the Chebyshev functions of any real order are also introduced and compared with other parallel concepts.

Keywords: periodic orbits, logistic map, modified Chebyshev polynomials.

2010 Mathematics Subject Classification: 37C25, 12E10.

Organization of the paper

The paper is divided into three main sections completed by references and three tables. The sections are:

1. **Introduction** – where besides the ideas and notations used in the paper also the investigated polynomials are introduced. Background of the paper subject-matter is presented as well.
2. **Boundary logistic map (with coefficient 4)** – in this section the periodic orbits of two classes of maps

$$g_w(x) = \frac{1}{w}\, x\,(4\,w - x), \qquad \text{and} \qquad h_w(x) = \frac{1}{w}\,(2\,w - x)^2$$

E. Hetmaniok, D. Słota, R. Wituła
Institute of Mathematics, Silesian University of Technology, Kaszubska 23, 44-100 Gliwice, Poland,
e-mail: {edyta.hetmaniok, damian.slota, roman.witula}@polsl.pl

P. Lorenc is the MSc graduate student in Mathematics
Faculty of Applied Mathematics, Silesian University of Technology, Kaszubska 23, 44-100 Gliwice, Poland, e-mail: plorenc@polsl.pl

R. Wituła, D. Słota, W. Hołubowski (eds.), *Monograph on the Occasion of 100th Birthday Anniversary of Zygmunt Zahorski.* Wydawnictwo Politechniki Śląskiej, Gliwice 2015, pp. 325–343.

are discussed. In the sequel we prove that both g_w and h_w possess 3-periodic orbits for every $w \in \mathbb{C}$, $w \neq 0$, which implies that g_w and h_w are chaotic in the Li-Yorke sense.

3. **New types of modified Chebyshev polynomials** – in this section on the grounds of discussion on the periodic orbits of polynomials g_w and h_w the new modified Chebyshev polynomials (perhaps the name "Chebyshev polynomials of the fifth kind" would be the most appropriate here) are defined
$$W_n(c^2(x)) := c^2(n\,x) = 4\,T_n^2\Big(\frac{c(x)}{2}\Big)$$
and
$$V_n(s^2(x)) := s^2(n\,x) = s^2(x)\,U_{n-1}^2\Big(\frac{c(x)}{2}\Big),$$
for every $n \in \mathbb{N}$, where $c(x) := 2\cos(x)$, $s(x) := 2\sin(x)$, $T_n(x)$ and $U_n(x)$ denote the n-th Chebyshev polynomials of the first and second kind, respectively (see [17, 21, 22, 29] for fundamental information about the Chebyshev polynomials). A number of basic properties of these polynomials are also presented, including the recurrence relations for W_n and V_n. Moreover, in Remark 3.2 we discuss some analytical generalizations of the Chebyshev polynomials which we intend to use for further investigations in a separate paper. The paper constitutes an essential supplement for discussion started by the authors in papers [23, 24, 31].

1. Introduction

In the paper we intend to discuss the periodic orbits of boundary logistic map. Main rôle in discussion will be played by the following polynomials (see [30, 33, 35, 37]):

$$p(\mathbb{X}) = \mathbb{X}^3 - 3\,\mathbb{X} + 1 = \prod_{k=0}^{2}\Big(\mathbb{X} - c(2^k\beta)\Big),$$

$$q(\mathbb{X}) = \mathbb{X}^3 + \mathbb{X}^2 - 2\,\mathbb{X} - 1 = \prod_{k=0}^{2}\Big(\mathbb{X} - c(2^k\alpha)\Big),$$

$$pc_2(\mathbb{X}) := p(\mathbb{X} - 2) = \mathbb{X}^3 - 6\,\mathbb{X}^2 + 9\,\mathbb{X} - 1 = \prod_{k=0}^{2}\Big(\mathbb{X} - c^2(2^k\beta)\Big),$$

$$ps_2(\mathbb{X}) := -p(2 - \mathbb{X}) = \mathbb{X}^3 - 6\,\mathbb{X}^2 + 9\,\mathbb{X} - 3 = \prod_{k=0}^{2}\Big(\mathbb{X} - s^2(2^k\beta)\Big),$$

$$qc_2(\mathbb{X}) := q(\mathbb{X} - 2) = \mathbb{X}^3 - 5\,\mathbb{X}^2 + 6\,\mathbb{X} - 1 = \prod_{k=0}^{2}\Big(\mathbb{X} - c^2(2^k\alpha)\Big),$$

$$qs_2(\mathbb{X}) := -q(2 - \mathbb{X}) = \mathbb{X}^3 - 7\,\mathbb{X}^2 + 14\,\mathbb{X} - 7 = \prod_{k=0}^{2}\Big(\mathbb{X} - s^2(2^k\alpha)\Big).$$

where $\alpha := \frac{2\pi}{7}$, $\beta := \frac{2\pi}{9}$. Furthermore $\gamma := \frac{2\pi}{11}$, $\delta := \frac{2\pi}{13}$, $\varepsilon := \frac{2\pi}{15}$ and

$$c(x) := 2\cos(x) \qquad \text{and} \qquad s(x) := 2\sin(x).$$

The above notation will be applied throughout the entire paper. In this connection we obtain the following form of the known trigonometric identities

$$\begin{aligned} &c(2\,x) = c^2(x) - 2 = 2 - s^2(x),\\ &s(2\,x) = s(x)\,c(x),\\ &s(x)\prod_{k=0}^{N-1} c(2^k x) = s(2^N x). \end{aligned}$$

We also use the notation

$$-A := \{-a :\ a \in A\}$$

for every nonempty $A \subset \mathbb{R}$.

Certainly, the problem of analytic description of such orbits in specific cases is still interesting for us. It turned out that there exist several particular quadratic and cubic polynomials, in case of which the analysis of their orbits is especially interesting because it is connected, among others, with determination of some new sequences of integers (associated with the length of appropriate orbits of given polynomials) and new sequences of “modified” Chebyshev polynomials possessing the intriguing properties. It seems that the obtained results are worth to be popularized for the sake of their accessibility and creativity. This paper is a substantial suplement of paper [32].

We say that n-periodic real orbit $\{x_1, x_2, \ldots, x_n\}$ of polynomial $p \in \mathbb{R}[x]$ possesses *the trigonometric form* if there exist $\alpha_1, \alpha_2, \ldots, \alpha_n \in [0, 2\pi)$ such that $x_i = c(\alpha_i)$ for every $i = 1, 2, \ldots, n$. In particular, if there exist $\alpha \in (0, 2\pi)$ and $k \in \mathbb{N}$ such that $x_i = c^2(k^i\,\alpha)$ or $x_i = s^2(k^i\,\alpha)$, respectively, for every $i = 1, 2, \ldots, n$, then orbit $\{x_1, x_2, \ldots, x_n\}$ will be called the *n-periodic square trigonometric orbit.*

Remark 1.1. All the above four decompositions can be deduced from respective connections with $p(\mathbb{X})$ and $q(\mathbb{X})$ (see also [29, 30]).

Remark 1.2. Polynomial $qs_2(\mathbb{X})$ is called the Johannes Kepler polynomial (see [29] volume III). We have $\mathbb{X}\,qs_2(\mathbb{X}) = 2\,T_7(\mathbb{X}/2)$ where $T_7(\mathbb{X})$ is the seventh Chebyshev polynomial of the first kind.

Remark 1.3. Let us also notice the unusual similarity between forms of coefficients of polynomial $qs_2(\mathbb{X})$ and the following one

$$\begin{aligned} \mathbb{X}^3 - 7\,\mathbb{X}^2 + 7\,\mathbb{X} + 7 &= \prod_{k=0}^{2}\Big(\mathbb{X} - \sqrt{7}\cot(2^k\alpha)\Big) =\\ &= \prod_{k=0}^{2}\Big(\mathbb{X} - 3 - 2\,c(2^k\alpha)\Big). \end{aligned}$$

Sums of the powers of roots of the above polynomial is described by sequence A215575 in Sloane’s OEIS.

One more polynomial "similar" to polynomial $qs_2(\mathbb{X})$ will appear in this paper. On the occasion of discussing the 4-elements orbits rescaling the limit logistic polynomial $g_w(x)$ we will deal with polynomial (see [29] volume I):

$$\mathbb{X}^4 - 7\,\mathbb{X}^3 + 14\,\mathbb{X}^2 - 8\,\mathbb{X} + 1 = \prod_{k=0}^{3} \left(\mathbb{X} - s^2\left(2^k \frac{\pi}{15}\right)\right) =$$

$$= \prod_{1\leqslant k,l\leqslant 2} \left(\mathbb{X} - \frac{1}{4}(7 + (-1)^k\sqrt{5} + (-1)^{k+l}\sqrt{30 + (-1)^k 6\sqrt{5}})\right).$$

Remark 1.4. In paper [9] the authors have proven that a real analytic function, whose Maclaurin series expansion has the form $\sum\limits_{n=0}^{\infty} a_n x^n$, behaves chaotically whenever the following condition holds

$$a_2^2 - a_1 a_3 > 0. \tag{1}$$

It can be easily observed that all the polynomials presented in this section, and simultaneously the all ones discussed in this paper, satisfy condition (1), i.e. they are chaotic.

2. Boundary logistic map (with coefficient 4)

Maps, discussed in this section (functions g_w and h_w defined in theorem given below), have a close connection with the boundary logistic map $x \mapsto 4\,x(1-x)$ [7]. Let us notice that if we replace coefficient 4 by number $a > 4$ then the obtained logistic map is chaotic [11]. Furthermore, this boundary logistic map has no attracting periodic points.

In the following theorem we present all the $n-$periodic orbits of functions g_w and h_w for $n = 1, 2, \mathbf{3}, 4, 5, 6$. We note that since both functions g_w and h_w possess the 3-periodic orbits, these functions possess also the $n-$periodic orbits for every $n \in \mathbb{N}$ (by Sharkovsky's Theorem). Unfortunately, we do not know the description of all these orbits for other $n \in \mathbb{N}$. We only suppose that for every positive integer $n \geq 7$ both g_w and h_w possess exclusively the $n-$periodic square trigonometric orbits of the form

$$\{s^2(2^k x) : \quad k = 0, 1, ..., n-1\},$$

where $x = x(n)$ is a rational number (probably $x = \frac{r}{2^n \pm 1}$, $(r, 2^n \pm 1) = 1$), and

$$\{c^2(2^k y) : \quad k = 0, 1, ..., n-1\},$$

where $y = y(n)$ is also a rational number (probably $y = \frac{t}{2^n \pm 1}$, $(t, 2^n \pm 1) = 1$), respectively.

Theorem 2.1. *Let* $g_w(x) := \frac{1}{w}x(4w-x)$ *and* $h_w(x) := \frac{1}{w}(2w-x)^2$, *where* $w \in \mathbb{C}\backslash\{0\}$. *Then we have* $g_w([0, 4w]) = [0, 4w]$ *and* $h_w([0, 4w]) = [0, 4w]$, *for every* $w \in \mathbb{C}$, *and the following identities hold*

$$g_w(w\,s^2(x)) = w\,s^2(2x), \tag{2}$$
$$h_w(w\,c^2(x)) = w\,c^2(2x). \tag{3}$$

In the sequel, from these relations we deduce that sets

$$\{w\,s^2(\alpha),\, w\,s^2(2\alpha),\, w\,s^2(4\alpha)\} \quad \textit{and} \quad \{w\,s^2(\beta),\, w\,s^2(2\beta),\, w\,s^2(4\beta)\}$$

are the 3*-periodic orbits of* g_w *and that sets*

$$\{w\,c^2(\alpha),\, w\,c^2(2\alpha),\, w\,c^2(4\alpha)\} \quad \textit{and} \quad \{w\,c^2(\beta),\, w\,c^2(2\beta),\, w\,c^2(4\beta)\}$$

are the 3*-periodic orbits of* h_w*. In both cases these are the unique* 3*−periodic orbits of these functions.*

Hence, from Sharkovsky's Theorem [3,5,6,15,19] we obtain that g_w *and* h_w *possess the periodic orbits of any finite cardinality.*

We note that the given above sets are the only possible 3*-periodic orbits of* g_w *and* h_w*, respectively. Moreover, numbers* 0 *and* $3w$ *are the only fixed points of* $g_w(x)$*, whereas numbers* w *and* $4w$ *are the only fixed points of* $h_w(x)$*, sets*

$$\{w\,s^2\left(2^k\frac{\pi}{5}\right) :\, k=0,1\} = \{w\left(\frac{5-\sqrt{5}}{2}\right), w\left(\frac{5+\sqrt{5}}{2}\right)\}$$

and

$$\{w\,c^2\left(2^k\frac{\pi}{5}\right) :\, k=0,1\} = \{w\left(\frac{\sqrt{5}-1}{2}\right)^2, w\left(\frac{\sqrt{5}+1}{2}\right)^2\}$$

are the only 2-periodic orbits of g_w *and* h_w*, respectively (we note that* $\sqrt{5}\,c\left(2^k\frac{\pi}{5}\right) = s^2\left(2^{k+1}\frac{\pi}{5}\right)$ *for every* $k=0,1$*), sets*

$$\{w\,s^2(2^k\gamma) :\, k=0,1,\ldots,4\} \quad \textit{and} \quad \{w\,c^2(2^k\gamma) :\, k=0,1,\ldots,4\}$$

are the 5*-periodic orbits of* g_w *and* h_w*, respectively (remaining* 5*-periodic orbits of* h_w *are presented in Remark 2.4), sets*

$$\{w\,s^2(2^k\delta) :\, k=0,1,\ldots,5\} \quad \textit{and} \quad \{w\,c^2(2^k\delta) :\, k=0,1,\ldots,5\}$$

are the 6*-periodic orbits of* g_w *and* h_w*, respectively, and at last, sets*

$$\{w\,s^2(2^k\varepsilon) :\, k=0,1,2,3\} = \{\frac{w}{4}\big(7+\sqrt{5}\pm\sqrt{6(5+\sqrt{5})}\big), \frac{w}{4}\big(7-\sqrt{5}\pm\sqrt{6(5-\sqrt{5})}\big)\}$$

and

$$\{w\,c^2(2^k\varepsilon) :\, k=0,1,2,3\} = \{\frac{w}{4}\big(9+\sqrt{5}\pm\sqrt{6(5-\sqrt{5})}\big), \frac{w}{4}\big(9-\sqrt{5}\pm\sqrt{6(5+\sqrt{5})}\big)\}$$

are the 4*-periodic orbits of* g_w *and* h_w*, respectively.*

Function g_w possesses additionally two other 4-periodic orbits:

$$\left\{w\, s^2\Big(2^k \frac{\pi}{17}\Big) : \, k = 1,2,3,4\right\} \quad \textit{and} \quad \left\{w\, s^2\Big(2^k \frac{3\pi}{17}\Big) : \, k = 0,1,2,3\right\}$$

(all these eight numbers for $w = 1$ are roots of the following polynomial

$$x^8 - 17\,x^7 + 119\,x^6 - 442\,x^5 + 935\,x^4 - 1122\,x^3 + 714\,x^2 - 204\,x + 17).$$

Similarly, function h_w possesses as well two other 4-periodic orbits:

$$\left\{w\, c^2\Big(2^k \frac{\pi}{17}\Big) : \, k = 1,2,3,4\right\} \quad \textit{and} \quad \left\{w\, c^2\Big(2^k \frac{3\pi}{17}\Big) : \, k = 0,1,2,3\right\}$$

(all these eight numbers for $w = 1$ form the set of zeros of polynomial

$$x^8 - 15\,x^7 + 91\,x^6 - 286\,x^5 + 495\,x^4 - 462\,x^3 + 210\,x^2 - 36\,x + 1).$$

Proof. Proof of the second part of theorem results from equalities

$$s^2(8\alpha) = s^2(\alpha) \quad \text{and} \quad s^2(8\beta) = s^2(\beta)$$

(we note that

$$s^2(\gamma) \overset{g_w}{\to} s^2(2\gamma) \overset{g_w}{\to} s^2(4\gamma) \overset{g_w}{\to} s^2(3\gamma) \overset{g_w}{\to} s^2(5\gamma) \overset{g_w}{\to} s^2(\gamma),$$
$$c^2(\varepsilon) \overset{h_w}{\to} c^2(2\varepsilon) \overset{h_w}{\to} c^2(4\varepsilon) \overset{h_w}{\to} c^2(7\varepsilon) \overset{h_w}{\to} c^2(\varepsilon).$$

etc.) and from the following decompositions

$$g_1 \circ g_1 \circ g_1(x) - x = -x\,(x-3)\, ps_2(x)\, qs_2(x),$$
$$h_1 \circ h_1 \circ h_1(x) - x = (x-1)\,(x-4)\, pc_2(x)\, qc_2(x),$$

since we have

$$g_w \circ g_w \circ g_w(w\,x) = w\, g_1 \circ g_1 \circ g_1(x)$$

and

$$h_w \circ h_w \circ h_w(w\,x) = w\, h_1 \circ h_1 \circ h_1(x).$$

□

Corollary 2.2. *For every positive integer $n \geq 3$ both g_w and h_w possess the $n-$periodic square trigonometric orbits of the form*

$$\{s^2\left(2^k \frac{\pi}{2^n+1}\right) : \, k = 0,1,...,n-1\},$$
$$\{s^2\left(2^k \frac{\pi}{2^n-1}\right) : \, k = 0,1,...,n-1\}$$

and

$$\{c^2\left(2^k \frac{\pi}{2^n+1}\right) : \, k = 0,1,...,n-1\},$$

$$\{c^2\left(2^k\frac{\pi}{2^n-1}\right) : \ k=0,1,...,n-1\},$$

respectively. Certainly, these functions may possess also the other n-periodic square trigonometric orbits which is especially interestingly exhibited in Remark 2.4.

Remark 2.3. Both maps $g_w\big|_{[0,4w]}$ and $h_w\big|_{[0,4w]}$ are transitive. It follows easily from the facts that both sequences $\left\{w\, s^2\big(2^n x\,\pi\big)\right\}_{n=0}^{\infty}$ and $\left\{w\, c^2\big(2^n x\,\pi\big)\right\}_{n=0}^{\infty}$ are dense in $[0,4w]$ for almost all $x\in\mathbb{R}$. It is a consequence of uniform distribution of the sequence $\left\{2^n x\right\}_{n=0}^{\infty}$ for almost all $x\in\mathbb{R}$ [12].

Remark 2.4. Polynomial $h_1(x)$ (and in consequence every $h_w(x)$) possesses additionally five other 5-periodic orbits:
– the first one:

$$x_1=c^2\Big(\frac{\pi}{33}\Big)=3.96386\xrightarrow{h_1}x_2=c^2\Big(\frac{2\pi}{33}\Big)=3.85674\xrightarrow{h_1}x_3=c^2\Big(\frac{4\pi}{33}\Big)=3.447483$$
$$\xrightarrow{h_1}x_4=c^2\Big(\frac{8\pi}{33}\Big)=2.09516\xrightarrow{h_1}x_5=s^2\Big(\frac{\pi}{66}\Big)=0.00905615,$$

– the second one:

$$x_1=c^2\Big(\frac{5\pi}{33}\Big)=3.16011\xrightarrow{h_1}x_2=c^2\Big(\frac{10\pi}{33}\Big)=1.34586\xrightarrow{h_1}x_3=c^2\Big(\frac{13\pi}{33}\Big)=0.427894$$
$$\xrightarrow{h_1}x_4=c^2\Big(\frac{7\pi}{33}\Big)=2.47152\xrightarrow{h_1}x_5=s^2\Big(\frac{5\pi}{66}\Big)=0.222329,$$

– the third one:

$$x_1=c^2\Big(\frac{\pi}{31}\Big)=3.95906\xrightarrow{h_1}x_2=c^2\Big(\frac{2\pi}{31}\Big)=3.83792\xrightarrow{h_1}x_3=c^2\Big(\frac{4\pi}{31}\Big)=3.37793$$
$$\xrightarrow{h_1}x_4=c^2\Big(\frac{8\pi}{31}\Big)=1.8987\xrightarrow{h_1}x_5=s^2\Big(\frac{\pi}{62}\Big)=0.0102614,$$

– the fourth one:

$$x_1=c^2\Big(\frac{3\pi}{31}\Big)=3.64153\xrightarrow{h_1}x_2=c^2\Big(\frac{6\pi}{31}\Big)=2.69461\xrightarrow{h_1}x_3=s^2\Big(\frac{7\pi}{62}\Big)=0.482484$$
$$\xrightarrow{h_1}x_4=c^2\Big(\frac{7\pi}{31}\Big)=2.30286\xrightarrow{h_1}x_5=s^2\Big(\frac{3\pi}{62}\Big)=0.0917215,$$

– the fifth one:

$$x_1=c^2\Big(\frac{5\pi}{31}\Big)=3.05793\xrightarrow{h_1}x_2=c^2\Big(\frac{10\pi}{31}\Big)=1.11921\xrightarrow{h_1}x_3=s^2\Big(\frac{9\pi}{62}\Big)=0.775788$$
$$\xrightarrow{h_1}x_4=c^2\Big(\frac{9\pi}{31}\Big)=1.49869\xrightarrow{h_1}x_5=s^2\Big(\frac{5\pi}{62}\Big)=0.251307.$$

Elements of the first and second orbit are zeros of polynomial

$$x^{10}-21\,x^9+188\,x^8-934\,x^7+2806\,x^6-5202\,x^5+$$
$$+\,5809\,x^4-3629\,x^3+1090\,x^2-120\,x+1,$$

Table 1

Orbits of g_1 including $\sin^2\left(\frac{\pi}{n}\right)$ for successive odd $n \geqslant 3$, where $l := \min\{k \in \mathbb{N} : \sin^2\left(2^k \frac{\pi}{n}\right) = \sin^2\left(\frac{\pi}{n}\right)\}$ denotes the length of every orbit and $N := \lfloor \frac{n}{l} \rfloor$ denotes the number of remaining orbits of g_1 generated by $\sin^2\left(\frac{r\pi}{n}\right)$, $(r,n)=1$

n	l	N	the orbit of g_1 including $\sin^2\left(\frac{\pi}{n}\right)$
3	1	3	$\left\{\frac{3}{4}\right\}$
5	2	2	$\left\{\frac{5}{8}-\frac{\sqrt{5}}{8}, \frac{5}{8}+\frac{\sqrt{5}}{8}\right\}$
7	3	2	$\left\{\sin^2\left(\frac{\pi}{7}\right), \cos^2\left(\frac{3\pi}{14}\right), \cos^2\left(\frac{\pi}{14}\right)\right\}$
9	3	3	$\left\{\sin^2\left(\frac{\pi}{9}\right), \sin^2\left(\frac{2\pi}{9}\right), \cos^2\left(\frac{\pi}{18}\right)\right\}$
11	5	2	$\left\{\sin^2\left(\frac{\pi}{11}\right), \sin^2\left(\frac{2\pi}{11}\right), \cos^2\left(\frac{3\pi}{22}\right), \cos^2\left(\frac{5\pi}{22}\right), \cos^2\left(\frac{\pi}{22}\right)\right\}$
13	6	2	$\left\{\sin^2\left(\frac{\pi}{13}\right), \sin^2\left(\frac{2\pi}{13}\right), \cos^2\left(\frac{5\pi}{26}\right), \cos^2\left(\frac{3\pi}{26}\right), \sin^2\left(\frac{3\pi}{13}\right), \cos^2\left(\frac{\pi}{26}\right)\right\}$
15	4	3	$\left\{\sin^2\left(\frac{\pi}{15}\right), \sin^2\left(\frac{2\pi}{15}\right), \cos^2\left(\frac{7\pi}{30}\right), \cos^2\left(\frac{\pi}{30}\right)\right\}$
17	4	4	$\left\{\sin^2\left(\frac{\pi}{17}\right), \sin^2\left(\frac{2\pi}{17}\right), \sin^2\left(\frac{4\pi}{17}\right), \cos^2\left(\frac{\pi}{34}\right)\right\}$
19	9	2	$\left\{\sin^2\left(\frac{\pi}{19}\right), \sin^2\left(\frac{2\pi}{19}\right), \sin^2\left(\frac{4\pi}{19}\right), \cos^2\left(\frac{3\pi}{38}\right), \sin^2\left(\frac{3\pi}{19}\right), \cos^2\left(\frac{7\pi}{38}\right), \cos^2\left(\frac{5\pi}{38}\right), \cos^2\left(\frac{9\pi}{38}\right), \cos^2\left(\frac{\pi}{38}\right)\right\}$
21	6	3	$\left\{\sin^2\left(\frac{\pi}{21}\right), \sin^2\left(\frac{2\pi}{21}\right), \sin^2\left(\frac{4\pi}{21}\right), \cos^2\left(\frac{5\pi}{42}\right), \sin^2\left(\frac{5\pi}{21}\right), \cos^2\left(\frac{\pi}{42}\right)\right\}$
23	11	2	$\left\{\sin^2\left(\frac{\pi}{23}\right), \sin^2\left(\frac{2\pi}{23}\right), \sin^2\left(\frac{4\pi}{23}\right), \cos^2\left(\frac{7\pi}{46}\right), \cos^2\left(\frac{9\pi}{46}\right), \cos^2\left(\frac{5\pi}{46}\right), \sin^2\left(\frac{5\pi}{23}\right), \cos^2\left(\frac{3\pi}{46}\right), \sin^2\left(\frac{3\pi}{23}\right), \cos^2\left(\frac{11\pi}{46}\right), \cos^2\left(\frac{\pi}{46}\right)\right\}$
25	10	2	$\left\{\sin^2\left(\frac{\pi}{25}\right), \sin^2\left(\frac{2\pi}{25}\right), \sin^2\left(\frac{4\pi}{25}\right), \cos^2\left(\frac{9\pi}{50}\right), \cos^2\left(\frac{7\pi}{50}\right), \cos^2\left(\frac{11\pi}{50}\right), \cos^2\left(\frac{3\pi}{50}\right), \sin^2\left(\frac{3\pi}{25}\right), \sin^2\left(\frac{6\pi}{25}\right), \cos^2\left(\frac{\pi}{50}\right)\right\}$
27	9	3	$\left\{\sin^2\left(\frac{\pi}{27}\right), \sin^2\left(\frac{2\pi}{27}\right), \sin^2\left(\frac{4\pi}{27}\right), \cos^2\left(\frac{11\pi}{54}\right), \cos^2\left(\frac{5\pi}{54}\right), \sin^2\left(\frac{5\pi}{27}\right), \cos^2\left(\frac{7\pi}{54}\right), \cos^2\left(\frac{13\pi}{54}\right), \cos^2\left(\frac{\pi}{54}\right)\right\}$
29	14	2	$\left\{\sin^2\left(\frac{\pi}{29}\right), \sin^2\left(\frac{2\pi}{29}\right), \sin^2\left(\frac{4\pi}{29}\right), \cos^2\left(\frac{13\pi}{58}\right), \cos^2\left(\frac{3\pi}{58}\right), \sin^2\left(\frac{3\pi}{29}\right), \sin^2\left(\frac{6\pi}{29}\right), \cos^2\left(\frac{5\pi}{58}\right), \sin^2\left(\frac{5\pi}{29}\right), \cos^2\left(\frac{9\pi}{58}\right), \cos^2\left(\frac{11\pi}{58}\right), \cos^2\left(\frac{7\pi}{58}\right), \sin^2\left(\frac{7\pi}{29}\right), \cos^2\left(\frac{\pi}{58}\right)\right\}$
31	5	6	$\left\{\sin^2\left(\frac{\pi}{31}\right), \sin^2\left(\frac{2\pi}{31}\right), \sin^2\left(\frac{4\pi}{31}\right), \cos^2\left(\frac{15\pi}{62}\right), \cos^2\left(\frac{\pi}{62}\right)\right\}$
33	5	6	$\left\{\sin^2\left(\frac{\pi}{33}\right), \sin^2\left(\frac{2\pi}{33}\right), \sin^2\left(\frac{4\pi}{33}\right), \sin^2\left(\frac{8\pi}{33}\right), \cos^2\left(\frac{\pi}{66}\right)\right\}$
35	12	2	$\left\{\sin^2\left(\frac{\pi}{35}\right), \sin^2\left(\frac{2\pi}{35}\right), \sin^2\left(\frac{4\pi}{35}\right), \sin^2\left(\frac{8\pi}{35}\right), \cos^2\left(\frac{3\pi}{70}\right), \sin^2\left(\frac{3\pi}{35}\right), \sin^2\left(\frac{6\pi}{35}\right), \cos^2\left(\frac{11\pi}{70}\right), \cos^2\left(\frac{13\pi}{70}\right), \cos^2\left(\frac{9\pi}{70}\right), \cos^2\left(\frac{17\pi}{70}\right), \cos^2\left(\frac{\pi}{70}\right)\right\}$

whereas the elements of the other three orbits are zeros of polynomial

$$\begin{aligned}
x^{15} &- 29\,x^{14} + 378\,x^{13} - 2925\,x^{12} + 14950\,x^{11} - 53130\,x^{10} + \\
&+ 134596\,x^{9} - 245157\,x^{8} + 319770\,x^{7} - 293930\,x^{6} + 184756\,x^{5} - \\
&- 75582\,x^{4} + 18564\,x^{3} - 2380\,x^{2} + 120\,x - 1.
\end{aligned}$$

Properties (2) and (3) for the cubic polynomials

We give here a description of all the cubic polynomials satisfying equalities (2) and (3) for given $x = \alpha$ and separately for given $x = \beta$.

Theorem 2.5. *Let* $r(x) = x^3 + a\,x^2 + b\,x + c$, $a, b, c \in \mathbb{C}$.

a) If

$$r\big(w\,s^2(2^k\alpha)\big) = w\,s^2(2^{k+1}\alpha), \tag{4}$$

for every $k = 0, 1, 2$, *where* $w \in \mathbb{C} \setminus \{0\}$, *then*

$$r(x) = x^3 - \Big(7\,w + \frac{1}{w}\Big)\,x^2 + (14\,w^2 + 4)\,x - 7\,w^3 = w^3 q s_2\big(\frac{x}{w}\big) + g_w(x).$$

Polynomial $r(x)$ *is the only one which satisfies condition (4). Set*

$$\{w\,s^2(\alpha),\ w\,s^2(2\alpha),\ w\,s^2(4\alpha)\}$$

is the 3*-periodic orbit of* $r(x)$ *and* $r(x)$ *possesses the* n*-periodic orbits for every* $n \in \mathbb{N}$.

b) If

$$r\big(w\,c^2(2^k\alpha)\big) = w\,c^2(2^{k+1}\alpha), \tag{5}$$

for every $k = 0, 1, 2$, *where* $w \in \mathbb{C} \setminus \{0\}$, *then*

$$r(x) = x^3 + \Big(-5\,w + \frac{1}{w}\Big)\,x^2 + (6\,w^2 - 4)\,x + 4\,w - w^3 = w^3 q c_2\big(\frac{x}{w}\big) + h_w(x).$$

Polynomial $r(x)$ *is the only one which satisfies condition (5). Set*

$$\{w\,c^2(\alpha),\ w\,c^2(2\alpha),\ w\,c^2(4\alpha)\}$$

is the 3*-periodic orbit of* $r(x)$ *and* $r(x)$ *possesses the* n*-periodic orbits for every* $n \in \mathbb{N}$.

c) If

$$r\big(w\,s^2(2^k\beta)\big) = w\,s^2(2^{k+1}\beta), \tag{6}$$

for every $k = 0, 1, 2$, *where* $w \in \mathbb{C} \setminus \{0\}$, *then*

$$r(x) = x^3 - \Big(6\,w + \frac{4}{w}\Big)\,x^2 + (4 + 9\,w^2)\,x - 3\,w^3 = w^3 p s_2\big(\frac{x}{w}\big) + g_w(x).$$

Moreover, $r(x)$ *is the only polynomial which satisfies condition (6). Set*

$$\{w\,s^2(\beta),\ w\,s^2(2\beta),\ w\,s^2(4\beta)\}$$

is the 3*-periodic orbit of* $r(x)$ *and* $r(x)$ *possesses the periodic orbits of every finite cardinality.*

d) If

$$r\big(w\,c^2(2^k\beta)\big) = w\,c^2(2^{k+1}\beta), \tag{7}$$

for every $k = 0, 1, 2$, *where* $w \in \mathbb{C} \setminus \{0\}$, *then*

$$r(x) = x^3 + \Big(\frac{1}{w} - 6\,w\Big)\,x^2 + (9\,w^2 - 4)\,x + 4\,w - w^3 = w^3 pc_2\big(\frac{x}{w}\big) + h_w(x).$$

Polynomial $r(x)$ *is the only polynomial which satisfies condition (7). Set*

$$\{w\,c^2(\beta),\; w\,c^2(2\beta),\; w\,c^2(4\beta)\}$$

is the 3*-periodic orbit of* $r(x)$ *and* $r(x)$ *possesses the periodic orbits of every finite cardinality.*

Remark 2.6. In this case we may put a question whether the reduction of conditions (4)–(7) influences the description of polynomials $r(x)$?

3. New types of modified Chebyshev polynomials

Properties (2) and (3) of polynomials g_w and h_w lead in natural way to generating two new types of the sequences of polynomials $\{W_n(x)\}_{n=0}^{\infty}$ and $\{V_n(x)\}_{n=0}^{\infty}$ defined by conditions

$$W_n(c^2(x)) = c^2(n\,x) = 4\,T_n^2\Big(\frac{c(x)}{2}\Big) \tag{8}$$

and

$$V_n(s^2(x)) = s^2(n\,x) = s^2(x)\,U_{n-1}^2\Big(\frac{c(x)}{2}\Big), \tag{9}$$

for every $n \in \mathbb{N} \cup \{0\}$, where $T_n(x)$ and $U_n(x)$ denote the n-th Chebyshev polynomials of the first and second kind, respectively. Hence we get

$$W_n(x^2) = 4\,T_n^2\Big(\frac{x}{2}\Big),$$

for every $x \in \mathbb{R}$, and

$$V_n(x) = x\,U_{n-1}^2\Big(\sqrt{1 - \frac{x}{4}}\Big),$$

for every $x \in (-\infty, 4]$.

It turns out that these polynomials are closely connected with the modified Chebyshev polynomials $\Omega_n(x) := 2T_n\left(\frac{x}{2}\right)$ (see [32, 38]):

$$W_n(x^2) = \Omega_n^2(x) = \Omega_{2n}(x) + 2 \tag{10}$$

and

$$V_n(x^2) = (-1)^{n-1}\Omega_{2n}(x) + 2, \tag{11}$$

from which the following identities result

$$W_{2n-1}(t) = V_{2n-1}(t),$$

$$W_{2n}(t) + V_{2n}(t) = 4$$

and

$$W_{2n}^2(x^2) + V_{2n}^2(x^2) = 2\,\Omega_{4n}^2(x) + 8 = 2\,\Omega_{8n}(x) + 12$$

for every $t, x \in \mathbb{R}$. Hence we deduce

$$W_n(1) = W_n\Big(c^2\Big(\frac{\pi}{3}\Big)\Big) = c^2\Big(n\,\frac{\pi}{3}\Big),$$
$$V_n(1) = V_n\Big(s^2\Big(\frac{\pi}{6}\Big)\Big) = s^2\Big(n\,\frac{\pi}{6}\Big),$$

which implies

$$V_n(1) = \begin{cases} W_n(1) = c^2\big(n\frac{\pi}{3}\big) & \text{for odd } n, \\ s^2\big(\frac{n}{2}\cdot\frac{\pi}{3}\big) & \text{for even } n. \end{cases}$$

Obviously one can prove inductively that (10)⇒(8) and (11)⇒(9). Let us present the proof of implication (11)⇒(9).

Proof. From (11) we obtain

$$\begin{aligned}(-1)^{n-1}V_n(s^2(x)) &= \Omega_{2n}(s(x)) + 2(-1)^{n-1} = 2\,T_{2n}(\sin x) + 2(-1)^{n-1} = \\ &= 2\cos\Big(2\,n\big(\frac{\pi}{2} - x\big)\Big) + 2(-1)^{n-1} = 2(-1)^n\cos(2\,n\,x) + 2(-1)^{n-1} = \\ &= 2(-1)^n(1 - 2\sin^2(n\,x)) + 2(-1)^{n-1} = (-1)^{n-1}s^2(n\,x)\end{aligned}$$

which implies (9). □

We know that [32]:

$$\Omega_n(\theta + \theta^{-1}) = \theta^n + \theta^{-n},$$

for every $\theta \in \mathbb{C} \setminus \{0\}$, which implies two interesting relations

$$W_n\big((\theta + \theta^{-1})^2\big) \overset{(10)}{=} (\theta^n + \theta^{-n})^2 \tag{12}$$

and

$$V_n\big((\theta + \theta^{-1})^2\big) \overset{(11)}{=} (-1)^{n-1}(\theta^n - (-\theta)^{-n})^2. \tag{13}$$

Moreover, the following decompositions are proven in paper [32]:

$$\Omega_{2n-1}(x) - \Omega_{2n-1}(\theta + \theta^{-1}) = \prod_{k=0}^{2n-2}\big(x - \theta\,\xi^{2k} - \theta^{-1}\xi^{-2k}\big), \tag{14}$$

where $\xi := \exp(i\,\pi/(2n-1))$, and

$$(-1)^n\Omega_{2n}(i\,x) + \Omega_{2n}(\theta + \theta^{-1}) = \prod_{k=0}^{2n-1}\big(x - \theta\,\zeta^{2k+1} + \theta^{-1}\zeta^{-2k-1}\big), \tag{15}$$

where $\zeta := \exp(i\,\pi/(2n))$. It implies, among others, that

$$
\begin{aligned}
&(-1)^n\Big(W_n(-x^2) - W_n\big((\theta+\theta^{-1})^2\big)\Big) = (-1)^n\Big(\Omega_{2n}(i\,x) - \theta^{2n} - \theta^{-2n}\Big) = \\
&\quad = (-1)^n\Omega_{2n}(i\,x) - (i\,\theta)^{2n} - (i\,\theta)^{-2n} = (-1)^n\Omega_{2n}(i\,x) + (i\,\zeta\theta)^{2n} + (i\,\zeta\theta)^{-2n} = \\
&\quad = \prod_{k=0}^{2n-1}\Big(x - i\big(\theta\zeta^{2k+2} + \theta^{-1}\zeta^{-2k-2}\big)\Big), \qquad (16)
\end{aligned}
$$

or in "positive" version ($x \mapsto i\,x$):

$$
W_n(x^2) - W_n\big((\theta+\theta^{-1})^2\big) = \prod_{k=0}^{2n-1}\Big(x - \theta\zeta^{2k+2} - \theta^{-1}\zeta^{-2k-2}\Big). \qquad (17)
$$

Similarly we determine

$$
\begin{aligned}
&-V_n(-x^2) + V_n\big((\theta+\theta^{-1})^2\big) = (-1)^n\Omega_{2n}(i\,x) - (-1)^n\Omega_{2n}(\theta+\theta^{-1}) = \\
&= (-1)^n\Omega_{2n}(i\,x) - (-1)^n(\theta^{2n}+\theta^{-2n}) = (-1)^n\Omega_{2n}(i\,x) + (i\,\zeta\theta)^{2n} + (i\,\zeta\theta)^{-2n} = \\
&\quad = \prod_{k=0}^{2n-1}\Big(x - i\,\theta\zeta^{2k+2} - i\,\theta^{-1}\zeta^{-2k-2}\Big). \qquad (18)
\end{aligned}
$$

Recurrence relations for polynomials W_n and V_n

The following recurrence relation for $\Omega_n(x)$ holds (see [30, 32]):

$$
\Omega_{n+2}(x) = x\,\Omega_{n+1}(x) - \Omega_n(x)
$$

which implies the two steps recurrence relation [14]:

$$
\Omega_{n+4}(x) = (x^2-2)\,\Omega_{n+2}(x) - \Omega_n(x). \qquad (19)
$$

Hence we deduce that

$$
\Omega_{2(n+2)}(x) + 2 = (x^2-2)(\Omega_{2(n+1)}(x)+2) - (\Omega_{2n}(x)+2) - 2\,x^2 + 8,
$$

i.e., by (10),

$$
W_{n+2}(t) = (t-2)\,W_{n+1}(t) - W_n(t) - 2\,t + 8 \qquad (20)
$$

which is the recurrence relation for $W_n(t)$. The first eleven polynomials W_n are presented in Table 2.

Similarly, from (19) one can conclude the relation

$$
\begin{aligned}
\Omega_{2(n+2)} + 2(-1)^{n+1} = (x^2-2)(\Omega_{2(n+1)}(x) + 2(-1)^n) - \\
- (\Omega_{2n}(x) + 2(-1)^{n-1}) - 2(-1)^n x^2,
\end{aligned}
$$

which generates the recurrence relation for $V_n(t)$:

$$
(-1)^{n+1}V_{n+2}(t) = (t-2)(-1)^n V_{n+1}(t) - (-1)^{n-1}V_n(t) - 2\,t\,(-1)^n,
$$

Table 2

Polynomials W_n

n	$W_n(t)$
0	4
1	t
2	$4-4t+t^2$
3	$9t-6t^2+t^3$
4	$4-16t+20t^2-8t^3+t^4$
5	$25t-50t^2+35t^3-10t^4+t^5$
6	$4-36t+105t^2-112t^3+54t^4-12t^5+t^6$
7	$49t-196t^2+294t^3-210t^4+77t^5-14t^6+t^7$
8	$4-64t+336t^2-672t^3+660t^4-352t^5+104t^6-16t^7+t^8$
9	$81t-540t^2+1386t^3-1782t^4+1287t^5-546t^6+135t^7-18t^8+t^9$
10	$4-100t+825t^2-2640t^3+4290t^4-4004t^5+2275t^6-800t^7+170t^8-20t^9+t^{10}$

Table 3

Polynomials V_n

n	$V_n(t)$
0	0
1	t
2	$4t-t^2$
3	$9t-6t^2+t^3$
4	$16t-20t^2+8t^3-t^4$
5	$25t-50t^2+35t^3-10t^4+t^5$
6	$36t-105t^2+112t^3-54t^4+12t^5-t^6$
7	$49t-196t^2+294t^3-210t^4+77t^5-14t^6+t^7$
8	$64t-336t^2+672t^3-660t^4+352t^5-104t^6+16t^7-t^8$
9	$81t-540t^2+1386t^3-1782t^4+1287t^5-546t^6+135t^7-18t^8+t^9$
10	$100t-825t^2+2640t^3-4290t^4+4004t^5-2275t^6+800t^7-170t^8+20t^9-t^{10}$

i.e.,

$$V_{n+2}(t) = (2-t)\,V_{n+1}(t) - V_n(t) + 2\,t. \tag{21}$$

The first eleven polynomials V_n are given in Table 3.

Let us also notice that polynomials $W_n(t)$ and $V_n(t)$ satisfy many identities of trigonometric nature compatible with the respective identities of standard trigonometry. For example, the following one's hold

$$
\begin{aligned}
&W_n(c^2(x)) + V_n(s^2(x)) = 4,\\
&W_n(c^2(x))\, V_n(s^2(x)) = V_{2n}(s^2(x)),\\
&(W_n(c^2(x)))^2 + (V_n(s^2(x)))^2 = 1 - 2V_{2n}(s^2(x)),\\
&V_n(s^2(x)) \prod_{k=0}^{N} W_{2^k n}(c^2(x)) = V_{2^{N+1}n}(s^2(x)).
\end{aligned}
$$

In particular, we obtain from this the following interesting numerical relations

$$
\begin{aligned}
&\lim_{x\to 0^+} \frac{V_{2^{N+1}n}(x)}{V_n(x)} = \prod_{k=0}^{N} W_{2^k n}(1),\\
&V_n(2) \prod_{k=0}^{N} W_{2^k n}(2) = V_{2^{N+1}n}(2),\\
&W_n(2) + V_n(2) = 4,\\
&2V_{2n}(2) = 1 - (W_n(2))^2 - (V_n(2))^2,\\
&W_n(0) + V_n(1) = W_n(1) + V_n(0) = 4,
\end{aligned}
$$

which implies

$$
\begin{aligned}
&V_n(0) = 4 - W_n(1) = s^2\big(n\frac{\pi}{3}\big),\\
&W_n(0) = 4 - V_n(1) = \begin{cases} s^2\big(n\frac{\pi}{3}\big) & \text{for odd } n,\\ c^2\big(n\frac{\pi}{6}\big) & \text{for even } n,\end{cases}\\
&\lim_{x\to 0^+} \frac{V_{2^{N+1}n}(x)}{V_n(x)} = \prod_{k=0}^{N} c^2\big(2^k n\frac{\pi}{3}\big) = \Big(\sin\big(2^{N+1}n\frac{\pi}{3}\big)/\sin\big(n\frac{\pi}{3}\big)\Big)^2.
\end{aligned}
$$

Moreover, we get

$$
\begin{aligned}
W_n(2) &= W_n\Big(c^2\Big(\frac{\pi}{4}\Big)\Big) = c^2\Big(n\frac{\pi}{4}\Big) = 2 + ((-1)^n + 1)(-1)^{\lceil n/2\rceil},\\
V_n(2) &= V_n\Big(s^2\Big(\frac{\pi}{4}\Big)\Big) = s^2\Big(n\frac{\pi}{4}\Big) = 2 - ((-1)^n + 1)(-1)^{\lceil n/2\rceil},\\
W_n(3) &= W_n\Big(c^2\Big(\frac{\pi}{6}\Big)\Big) = c^2\Big(n\frac{\pi}{6}\Big),\\
V_n(3) &= V_n\Big(s^2\Big(\frac{\pi}{3}\Big)\Big) = s^2\Big(n\frac{\pi}{3}\Big),
\end{aligned}
$$

which implies

$$
\begin{aligned}
&W_n(3) = 1 - V_n(1), \qquad\qquad V_n(3) = 1 - W_n(1),\\
&W_n(4) = W_n\big(c^2(0)\big) = 4,\\
&V_n(4) = V_n\Big(s^2\Big(\frac{\pi}{2}\Big)\Big) = s^2\Big(n\frac{\pi}{2}\Big) = \begin{cases} 0 & \text{for even } n,\\ 4 & \text{for odd } n,\end{cases}
\end{aligned}
$$

and so on. Simultaneously, from (12) we can deduce that

$$W_n(5) = L_{2n} + 2,$$

where L_n denote the n-th Lucas number, since

$$\theta + \theta^{-1} = \sqrt{5} \quad \Longleftrightarrow \quad \theta_1 = \frac{\sqrt{5}+1}{2} \quad \text{or} \quad \theta_2 = \frac{\sqrt{5}-1}{2}$$

and by Binet's formula for L_n we have

$$\theta_i^{2n} + \theta_i^{-2n} = L_{2n},$$

for every $i = 1, 2$, and $n \in \mathbb{N}$.

Generally we get

$$W_n(4k) = \big(\sqrt{k} + \sqrt{k-1}\big)^{2n} + 2 + \big(\sqrt{k} - \sqrt{k-1}\big)^{-2n}$$

for $k \in \mathbb{C}$, which implies

$$W_n\Big(4\,\frac{c^2}{b^2}\Big) = \Big(\frac{c+a}{b}\Big)^{2n} + 2 + \Big(\frac{c-a}{b}\Big)^{-2n}$$

for $a, b, c \in \mathbb{C}$, $b \neq 0$, and $a^2 + b^2 = c^2$. For example, we obtain

$$W_n\Big(\Big(\frac{13}{6}\Big)^2\Big) = \big(5^n + 5^{-n}\big)^2$$

and

$$W_n\Big(\Big(\frac{26}{5}\Big)^2\Big) = \Big(\Big(\frac{3}{2}\Big)^n + \Big(\frac{2}{3}\Big)^n\Big)^2,$$

whenever $a = 5$, $b = 12$, $c = 13$.

At last, let us observe that if

$$\Big(\frac{\sqrt{5}+1}{2}\Big)^{2^k} = \frac{a_k + b_k\sqrt{5}}{2}, \qquad a_k, b_k, k \in \mathbb{N},$$

then

$$W_n\big(a_k^2\big) = L_{2^k n} + 2.$$

For example, $W_n(9) = L_{4n} + 2$, $W_n(49) = L_{8n} + 2$, $W_n(2209) = L_{16n} + 2$.

Proof. If $\big(\frac{\sqrt{5}+1}{2}\big)^{2^k} = \frac{a_k + b_k\sqrt{5}}{2}$ then $a_k, b_k \in 2\mathbb{N} - 1$ and $\big(\frac{\sqrt{5}-1}{2}\big)^{2^k} = \frac{a_k - b_k\sqrt{5}}{2}$. It implies $a_k^2 - 5\,b_k^2 = 4$ and $\theta^2 - a_k\,\theta + 1 = 0$ which is equivalent to $\theta = \frac{a_k \pm b_k\sqrt{5}}{2}$. $\square$

We note also that from (17) we obtain

$$W_n(0) - W_n\big((\theta+\theta^{-1})^2\big) = \prod_{k=1}^{2n}\big(\theta\zeta^{2k} + \theta^{-1}\zeta^{-2k}\big).$$

Remark 3.1. From relations (8) and (9) it follows that polynomials $V_n(x)$, $n \in \mathbb{N} \cup \{0\}$, and separately $W_n(x)$, $n \in \mathbb{N} \cup \{0\}$, are not orthogonal on any nontrivial interval $I \subset \mathbb{R}$.

However, from (10) and (11) we deduce the following orthogonal relations

$$\int_{-2}^{2} \big(W_m(x^2) - 2\big)\big(W_n(x^2) - 2\big)(4 - x^2)^{-1/2} dx = N_m \delta_{m,n}$$

with $N_0 = 4\pi$ and $N_m = 2\pi$ if $m \neq 0$, and

$$\int_{-2}^{2} \big(V_m(x^2) - 2\big)\big(V_n(x^2) - 2\big)(4 - x^2)^{-1/2} dx = (-1)^{m+n} N_m \delta_{m,n}.$$

More properties of these polynomials, as well as their new applications, we intend to discuss in a separate paper.

Remark 3.2. (of conceptional nature) Next generalizations of the Chebyshev polynomials, from among many known ones (see [2]) together with the one discussed in the current paper, led us to one more kind of questions concerning the generalizations of these polynomials, this time to the real indices (and even the complex indices). For example

$$T_\xi(\cos x) = \cos(\xi\, x), \quad x \in \mathbb{R}, \quad \xi > 0.$$

We note that after simple algebra we get then the following differential equation

$$(1 - t^2)\, T_\xi''(t) - t\, T_\xi'(t) + \xi^2 T_\xi(t) = 0.$$

The idea of proposing that kind of generalization can arise from the connection of polynomial $T_n(x)$ with the hypergeometric functions and next with their integral representation (the latter is not necessary, however it enables to analyse better this generalization). Thus we have (see [1,18]):

$$T_n(x) = {}_2F_1\Big(-n, n; \frac{1}{2}; \frac{1-x}{2}\Big),$$

for every $x \in \mathbb{C}$, and

$${}_2F_1(\alpha, \beta; \gamma; x) = \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma - \beta)} \int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}(1 - t\, x)^{-\alpha} dt,$$

for $x, \alpha, \beta, \gamma \in \mathbb{C}$, $|\arg(1-x)| < \pi$ and $\mathrm{Re}(\gamma) > \mathrm{Re}(\beta) > 0$. By this facts we may take

$$T_\xi(x) := {}_2F_1\Big(-\xi, \xi; \frac{1}{2}; \frac{1-x}{2}\Big),$$

for every $x \in \mathbb{C}$ satisfying condition $\left|\frac{1-x}{2}\right| < 1$, and even

$$T_{\xi,\zeta}(x) := {}_2F_1\Big(-\xi, \zeta; \frac{1}{2}; \frac{1-x}{2}\Big),$$

for $\xi, \zeta \in \mathbb{C}$, $\mathrm{Re}(\xi) > 0$, $\frac{1}{2} > \mathrm{Re}(\zeta) > 0$, $\left|\arg\left(\frac{1+x}{2}\right)\right| < \pi$.

Another source of possible generalization of the Chebyshev polynomials can be found in the inspiring paper [27]. For example, starting from identity

$$T_n(\theta + \theta^{-1}) = \theta^n + \theta^{-n},$$

for $\theta \in \mathbb{C} \setminus \{0\}$, we can take either the first hyperbolic cosine formula

$$T_\xi(x) = x\Big(1 + \sum_{n=1}^{\infty} \frac{(x^2-1)^n}{(2n)!} \prod_{k=1}^{n} (\xi^2 - (2k-1)^2)\Big),$$

or the second hyperbolic cosine formula

$$T_\xi(x) = 1 + \xi^2(x^2-1)\Big(1 + 2\sum_{n=1}^{\infty} \frac{(2(x-1))^n}{(2n+2)!} \prod_{k=1}^{n} (\xi^2 - k^2)\Big).$$

Both formulae are compatible with our hypergeometric ones.

Some other type of generalization of the Chebyshev polynomials is discussed in [2] (see also [8]).

In the similar manner, i.e. by applying the hypergeometric function, the other classical orthogonal polynomials can also be generalized, for example the Legendre polynomials (see [16], page 56):

$$P_\xi(x) := {}_2F_1\Big(-\xi, \xi+1; 1; \frac{1}{2} - \frac{1}{2}x\Big),$$

for $-1 \leqslant x \leqslant 1$, which generates the excellent connections with the classical Legendre polynomials

$$P_\xi(x) = \frac{\sin \xi\pi}{\pi} \sum_{n=0}^{\infty} (-1)^n \Big[\frac{1}{\xi - n} - \frac{1}{\xi + n + 1}\Big] P_n(x),$$

for $\xi \in \mathbb{R} \setminus \mathbb{Z}$, $x \in (-1, 1]$, and

$$P_\xi(\cos\theta) P_\xi(\cos\theta') = \frac{\sin \xi\pi}{\pi} \sum_{n=0}^{\infty} (-1)^n \Big[\frac{1}{\xi - n} - \frac{1}{\xi + n + 1}\Big] P_n(\cos\theta) P_n(\cos\theta').$$

Bibliography

1. Beals R., Wong R.: *Special Functions*. Cambridge University Press, Cambridge 2010.
2. Bojanov B.D., Rahman Q.J.: *On certain extremal problems for polynomials*. J. Math. Anal. Appl. **189** (1995), 781–800.
3. Ciesielski K, Pogoda Z.: *On ordering the natural numbers or the Sharkovski Theorem*. Amer. Math. Monthly **115** (2008), 159–165.

4. Darst R.B., Palagallo J.A., Price T.E.: *Curious Curves*. World Scientific, New Jersey 2010.
5. Devaney R.L.: *An Introduction to Chaotic Dynamical Systems*. Addison-Wesley, Boston 1989.
6. Du B.S.: *A simple proof of Sharkovsky's Theorem revisited*. Amer. Math. Monthly **114** (2007), 152–155.
7. Elaydi S.N.: *An Introduction to Difference Equations*. Springer, New York 2005.
8. García Ravelo J., Cuevas R., Queijeiro A., Peña J.J., Morales J.: *Chebyshev functions of half-integer order*. Integral Transforms Spec. Funct. **18** (2007), 743–749.
9. Hacibekiroğlu G., Cağlar M., Polatoğlu Y.: *The higher order schwarzian derivative: its applications for chaotic behavior and new invariant sufficient condition of chaos*. Nonlinear Anal. Real World Appl. **10** (2009), 1270–1275.
10. Hsu C.-H., Li M.-C.: *Transitivity implies period six: a simple proof*. Amer. Math. Monthly **109** (2002), 840–843.
11. Kraft R.L.: *Chaos, Cantor sets and hyperbolicity for the logistic maps*. Amer. Math. Monthly **106** (1999), 400–408.
12. Kuipers L., Niederreiter H.: *Uniform Distribution of Sequences*. Wiley, New York 1974.
13. Kwietniak D., Oprocha P.: *Chaos theory from the mathematical viewpoint*. Mat. Stosow. **9** (2008), 1–45 (in Polish).
14. Latushkin Y., Ushakov V.: *A representation of regular subsequences of recurrent sequences*. Fibonacci Quart. **43.1** (2005), 70–84.
15. Li T.Y., Yorke J.A.: *Period three implies chaos*. Amer. Math. Monthly **82** (1975), 985–992.
16. Magnus W., Oberhettinger F.: *Formeln und Sätze für die Speziellen Funktionen der Mathematischen Physik*. Springer, Berlin 1943.
17. Mason J.C., Handscomb D.C.: *Chebyshev Polynomials*. Chapman & Hall/CRC, Boca Raton 2003.
18. Mathai A.M., Haubold H.J.: *Special Functions for Applied Scientists*. Springer, New York 2008.
19. Misiurewicz M.: *Remarks on Sharkovsky's Theorem*. Amer. Math. Monthly **104** (1997), 846–847.
20. Peitgen H.-O., Jürgens H., Saupe D.: *Chaos and Fractals, New Frontiers of Science*. Springer, New York 1992.
21. Rivlin T.J.: *Chebyshev Polynomials from Approximation Theory to Algebra and Number Theory*. Wiley, New York 1990.
22. Paszkowski S.: *Numerical Applications of Chebyshev Polynomials and Series*. PWN, Warsaw 1975 (in Polish).
23. Trawiński T., Kołton W., Hetmaniok E., Słota D., Wituła R.: *Analysis of chaotic phenomena occurring in chosen branched kinematic chains of robot manipulators*. In: *Wybrane zagadnienia elektrotechniki i elektroniki*, WZEE 2013, Czarna (2013), 1–8.
24. Trawiński T., Kołton W., Hetmaniok E., Słota D., Wituła R.: *Chaotic phenomena occurring in chosen kinematic chains of robot manipulators* (in preparation).
25. Tričković S.B., Stanković M.S.: *On periodic solutions of a certain difference equation*. Fibonacci Quart. **42.4** (2004), 300–305.
26. Vellekoop M., Berglund R.: *On intervals, transitivity = chaos*. Amer. Math. Monthly **101** (1994), 353–355.
27. Verde-Star L., Srivastava H.M.: *Some binomial formulas of the generalized Appel form*. J. Math. Anal. Appl. **274** (2002), 755–771.
28. Wang Z.X., Guo D.R.: *Special Functions*. World Scientific, New Jersey 2010.
29. Wituła R.: *Complex Numbers, Polynomials and Partial Fraction Decomposition*, vol. I–III. Wyd. Pol. Śl., Gliwice 2010 (in Polish).
30. Wituła R.: *Formulae for Sums of Unimodular Complex Numbers*. WPKJS, Gliwice 2011 (in Polish).
31. Wituła R., Hetmaniok E., Słota D., Trawiński T., Kołton W.: *On the Three, Five and Other Periodic Orbits of Some Polynomials*. In: Analysis and Simulation of Electrical and Computer Systems, L. Gołębiowski and D. Mazur (eds.), Lecture Notes in Electrical Engineering **324** (2015), doi 10.1007/978-3-319-11248-0_8.
32. Wituła R., Słota D.: *On modified Chebyshev polynomials*. J. Math. Anal. Appl. **324** (2006), 321–343.
33. Wituła R., Słota D.: *New Ramanujan-type formulas and quasi-Fibonacci numbers of order 7*. J. Integer Seq. **10** (2007), article 07.5.6.

34. Wituła R., Słota D.: *Quasi-Fibonacci numbers of order 11.* J. Integer Seq. **10** (2007), article 07.8.5.
35. Wituła R., Słota D.: *Decomposition of certain symmetric functions of powers of cosine and sine functions.* Int. J. Pure Appl. Math. **50** (2009), 1–12.
36. Wituła R., Słota D.: *Fixed and periodic points of polynomials generated by minimal polynomials of* $2\cos(\pi/n)$. Internat. J. Bifur. Chaos Appl. Sci. Engrg. **19** (2009), 3005–3016.
37. Wituła R., Słota D.: *Quasi-Fibonacci numbers of order 13 on the occasion of the Thirteenth International Conference on Fibonacci Numbers and Their Applications.* Congr. Numer. **201** (2010), 89–107.
38. Wituła R., Słota D.: *Some identities for the moments of random variables having a linear rescaling the Student's t distribution.* J. Math. Anal. Appl. **361** (2010), 276–279.